Topic 5 —

Determinants

The determinant will allow us to detect when a square matrix has an inverse.

Def: Let A be an $n \times n$ matrix. The matrix A_{ij} is defined to be the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and j-th column from A.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

remove row 2, column 3

$$A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & \cancel{2} & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & \cancel{8} & 9 \end{pmatrix}$$

remove
row 2
column 2

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ 5 & 6 & \cancel{7} & 8 \\ 9 & 10 & \cancel{11} & 12 \\ 13 & 14 & \cancel{15} & 16 \end{pmatrix}$$

remove
row 1
column 3

Def: Let A be an $n \times n$ matrix. Let a_{ij} be the number in the i-th row and j-th column of A. Define the <u>determinant of A</u>, denoted by $\det(A)$, as follows:

1. If $n=1$ and $A = (a_{11})$ then $\det(A) = a_{11}$

2. If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

   then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

3. $\rightarrow$

3 If $n \geq 3$, then pick a column j (where $1 \leq j \leq n$) to "expand on" and define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{i \text{ sums over the rows of } A \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

This is called the expansion of the <u>determinant along the j-th column</u>

<u>Note:</u> In step 3, you can instead expand along a row i (where $1 \leq i \leq n$) and then

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{j \text{ sums over the columns of } A \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Note: One can show that it doesn't matter what column or row you pick to expand on. You'll always get the same determinant for A.

Notation: We will also use bars instead of det(A) for determinant. Like this:

$$\det\begin{pmatrix} 5 & -1 \\ 10 & \frac{1}{2} \end{pmatrix} = \begin{vmatrix} 5 & -1 \\ 10 & \frac{1}{2} \end{vmatrix}$$

Ex:

$$\det(10) = 10$$

Ex:

$$\det\begin{pmatrix} 1 & 5 \\ -2 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 \\ -2 & 7 \end{pmatrix}$$

$$= (1)(7) - (5)(-2)$$

$$= 7 + 10 = 17$$

Ex:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{array}{c} \begin{pmatrix} 3 & \boxed{1} & 0 \\ -2 & \boxed{-4} & 3 \\ 5 & \boxed{4} & -2 \end{pmatrix} \\ \uparrow \\ \boxed{j=2} \end{array}$$

Expand on column $j=2$

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} \det(A_{i2})$$

$$= \underbrace{(-1)^{1+2} a_{12} \det(A_{12})}_{i=1 \text{ term}}$$

$$+ \underbrace{(-1)^{2+2} a_{22} \det(A_{22})}_{i=2 \text{ term}}$$

$$+ \underbrace{(-1)^{3+2} a_{32} \det(A_{32})}_{i=3 \text{ term}}$$

$$\boxed{8}$$

$$= \underbrace{(-1)(1)\begin{vmatrix}-2 & 3\\ 5 & -2\end{vmatrix}}_{(-1)^{1+2}a_{12}\det(A_{12})} \leftarrow \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$+ \underbrace{(1)(-4)\begin{vmatrix}3 & 0\\ 5 & -2\end{vmatrix}}_{(-1)^{2+2}a_{22}\det(A_{22})} \leftarrow \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$+ \underbrace{(-1)(4)\begin{vmatrix}3 & 0\\ -2 & 3\end{vmatrix}}_{(-1)^{3+2}a_{32}\det(A_{32})} \leftarrow \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$= -\Big((-2)(-2)-(3)(5)\Big)$$

$$-4\Big((3)(-2)-(0)(5)\Big)$$

$$-4\Big((3)(3)-(0)(-2)\Big)$$

$$= -[-11]-4[-6]-4[9] = \boxed{-1}$$

$$\underbrace{\begin{pmatrix}(-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3}\\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3}\\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3}\end{pmatrix}}_{\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}}$$

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So we have that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

Ex:

we showed that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

by expanding on column 2.

We now expand on row 1 to see if we get the same answer.

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$$\boxed{\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= \overbrace{(3)}^{+3}\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + \overbrace{(0)}^{+0}\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= 3\left[(-4)(-2) - (3)(4)\right] - 1 \cdot \left[(-2)(-2) - (3)(5)\right] + 0$$

$$= 3[-4] - [-11] = -12 + 11 = -1$$

Ex:

$$\det\begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&-1\\2&1&3&0\\ \boxed{1\quad 0\quad 0\quad -2}\\1&1&2&0\end{pmatrix}$$

$$\begin{pmatrix}+&-&+&-\\-&+&-&+\\ \boxed{+\quad -\quad +\quad -}\\-&+&-&+\end{pmatrix}$$

expand on 3rd row

$$= \underbrace{(1)}_{+1}\cdot\begin{vmatrix}0&0&-1\\1&3&0\\1&2&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$\overbrace{-(0)}^{-0}\cdot\begin{vmatrix}1&0&-1\\2&3&0\\1&2&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$\overbrace{+(0)}^{+0}\cdot\begin{vmatrix}1&0&-1\\2&1&0\\1&1&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$\overbrace{-(-2)}^{-(-2)}\cdot\begin{vmatrix}1&0&0\\2&1&3\\1&1&2\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -1 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

expand on row 1 on both

$$= 0\cdot\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$+2\left[(1)\cdot\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 0\cdot\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$= -1\cdot\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -[2-3] + 2[2-3]$$

$$= -1$$

Fast way to do 3x3 determinants

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

take determinant of this

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{matrix}$$

multiply together and add

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{matrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{matrix}$$

multiply together and put a -1 in front of each one and add

Then add all results:

$$(1)(1)(0) + (-1)(2)(3) + (1)(0)(2)$$
$$-(1)(1)(3) - (1)(2)(2) - (-1)(0)(0)$$
$$= 0 - 6 + 0 - 3 - 4 + 0 = \boxed{-13}$$

Let's verify.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

$$= -0\cdot\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 2\cdot\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

$$= [0-3] - 2[2-(-3)] = -3-10 = \boxed{-13}$$

Properties of the determinant

Let A and B be $n \times n$ matrices.

1. $\det(A^T) = \det(A)$
2. $\det(AB) = \det(A) \cdot \det(B)$
3. If A^{-1} exists, then $\det(A) \neq 0$
4. If $\det(A) \neq 0$, then A^{-1} exists
5. If A^{-1} exists, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$
6. $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix
7. If A has a row of zeros or a column of zeros, then $\det(A) = 0$

Ex:

$$\det\begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 0 \\ 5 & 4 & 0 \end{pmatrix} = 0$$

column of zeros

8 If a row is a multiple of another row, or if a column is a multiple of another column in A, then $\det(A) = 0$.

Ex of 8:

$$\det\begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 2 \\ -1 & -3 & 7 \end{pmatrix} = 0$$

(column 2) = 3 · (column 1)

that is, column 2 is a multiple of column 1

Formula for A^{-1} for 2x2 matrix A

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then, $\det(A) = ad - bc$

If $\det(A) = ad - bc \neq 0$, then

A^{-1} exists and

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

det(A)

Ex: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$

So, A^{-1} exists and

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{-2} & \frac{-2}{-2} \\ \frac{-3}{-2} & \frac{1}{-2} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

det(A)